

Optimal Amplify-and-Forward Schemes for Relay Channels with Correlated Relay Noise

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Abstract—This paper investigates amplify-and-forward (AF) schemes for both one and two-way relay channels. Unlike most existing works assuming independent noise at the relays, we consider a more general scenario with correlated relay noise. We first propose an approach to efficiently solve a class of quadratically constrained fractional problems via second-order cone programming (SOCP). Then it is shown that the AF relay optimization problems studied in this paper can be incorporated into such quadratically constrained fractional problems. As a consequence, the proposed approach can be used as a unified framework to solve the optimal AF rate for the one-way relay channel and the optimal AF rate region for the two-way relay channel under both sum and individual relay power constraints.

In particular, for one-way relay channel under individual relay power constraints, we propose two suboptimal AF schemes in closed-form. It is shown that they are approximately optimal in certain conditions of interest. Furthermore, we find an interesting result that, on average, noise correlation is beneficial no matter the relays know the noise covariance matrix or not for such scenario. Overall, the obtained results recover and generalize several existing results for the uncorrelated counterpart.

I. INTRODUCTION

Since the amplify-and-forward (AF) relay scheme was introduced, it has been widely studied in the context of cooperative communication [1]–[3]. It is an interesting technique from the practical standpoint because the complexity and cost of relaying, always an issue in designing cooperative networks, are minimal for AF relay networks. As the simplest coding scheme, AF is also used to estimate the relay network capacity. Obviously, the achievable rate of the AF scheme can be viewed as a lower bound to the network capacity. In addition to its simplicity, AF is known to be the optimal relay strategy in many interesting cases, e.g., [4] and [5].

In certain resource constrained networks, collaborative AF schemes have been developed to exploit the multi-antenna gain in multiple relay networks. The problem of finding the optimal AF scheme for two-hop one-way relay channel under sum or/and individual relay power constraints has been extensively studied in previous works [6]–[9]. Under the sum power constraint, an algebraic approach was proposed to find the optimal AF scheme in a closed form in terms of maximizing the transmission rate [8], [9]. However, the problem becomes more challenging when the relays are subject to individual power constraints. For this problem, a semi-closed form solution has been derived in [6]. The authors in [7] also developed an iterative algorithm to solve the problem. In [11], the algebraic

approach was generalized to solve the sum/individual rate maximization problem for a two-hop multiple access relay channel. Moreover, the AF scheme design issues have been considered for more general layered [13]–[15] and non-layered [12] relay networks.

Recently, analog network coding (ANC) [16] extends the AF-based one-way relaying scheme to a two-way relay channel to support communications in two directions via a two-step protocol. The problem of finding the optimal rate region of a two-way relay channel with a two-step AF protocol has been studied for different network setups. The authors in [17] characterized the maximum achievable rate region for the two-way relay channel with a single multi-antenna relay; while the maximum rate region for the AF two-way relay channel with multiple single-antenna relays has been studied in [18] and [19] under both sum and individual relay power constraints.

The AF scheme designs mentioned above has been built upon the assumption of independent relay noise. However, in wireless networks, noise correlation between nodes may occur due to the common interference or noise propagation, and the previous design approaches cannot be directly applied to such correlated scenarios. For a two-hop relay network with noise correlation, it is worthwhile to mention the pioneer work [10], which presented a closed-form solution under the sum relay power constraint. With the individual relay power constraints, although the algorithm in [7] can be generalized to solve this problem, its convergence rate is not guaranteed theoretically.

In this paper, considering both sum and individual power constraints at the relays, our focus is on the design of optimal AF schemes for both one and two-way relay channels with correlated relay noise. Naturally, our design problem is more general and difficult than the existing ones. For simplicity, we assume that all the network parameters are real. Our main contributions are summarized as follows.

- 1) We study a class of quadratically constrained fractional problems which is generally non-convex. With the aid of some transformation tricks, we show that this class of problems can be recast as a collection of convex SOCP problems and thus can be efficiently solved via interior point methods. The reason why we study such problems is that they can be used as a unified framework for formulating the AF relay optimization problems for both one and two-way relay channels under different power constraints.

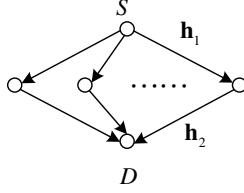


Fig. 1. One-way Relay Channel

- 2) We obtain the optimal AF scheme for one-way relay channel using the proposed formulation. Specifically, it is sufficient to get the optimal solution via solving only one SOCP problem. For individual power constraint, we also propose two suboptimal AF schemes which are approximately optimal under certain conditions of interest. Finally, we show that the noise correlation is beneficial to the AF performance on average.
- 3) We also study the two-way AF relay channel without the assumption of channel reciprocity. We apply the proposed formulation rather than the non-convex weighted sum rate maximization (WSRMax) problem to find the boundary point of the union rate region. In particular, each boundary point can be calculated by solving two SOCP problems.

Notation: $\text{diag}\{\mathbf{x}\}$ denotes a diagonal matrix with the elements in \mathbf{x} on its main diagonal and $\text{vec}\{\mathbf{X}\} = [x_1, \dots, x_n]^T$ with x_k on the diagonal of \mathbf{X} . We use $\mathcal{N}(\boldsymbol{\mu}, \mathbf{K})$ to denote n -dimensional joint Gaussian distribution with means $\boldsymbol{\mu}$ and covariance matrix \mathbf{K} . $\log(\cdot)$ denotes the logarithm in the base 2 and $\mathbb{E}[\cdot]$ denotes the expectation of a random variable.

II. A UNIFIED FRAMEWORK

As is shown in great many works, designing the optimal AF schemes in relay networks is formulated as an optimization problem. In this section, we provide a class of quadratically constrained fractional problems which can be used as a unified framework for formulating such problems for both one and two-way AF relay channels.

Let's consider the quadratically constrained fractional problem shown as follows.

$$\begin{aligned} \max_{\mathbf{x}} \quad & \frac{\mathbf{x}^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}}{\mathbf{x}^T \mathbf{A}_0 \mathbf{x} + c_0} \\ \text{s.t.} \quad & \mathbf{x}^T \mathbf{A}_i \mathbf{x} - \mathbf{x}^T \mathbf{b}_i \mathbf{b}_i^T \mathbf{x} + c_i \leq 0, \quad i = 1, \dots, m_1 \\ & \mathbf{x}^T \mathbf{A}_j \mathbf{x} + c_j \leq 0, \quad j = m_1 + 1, \dots, m_1 + m_2 \end{aligned} \quad (\text{P1})$$

where $c_i \geq 0$, $i = 0, 1, \dots, m_1$, and \mathbf{A}_i for $i = 0, \dots, m_1$ and \mathbf{A}_j for $j = m_1 + 1, \dots, m_1 + m_2$ are symmetric positive semi-definite matrices. It is observed that the objective function of the above problem is not concave and the first m_1 constraints are not convex. As a result, it is not a standard convex optimization problem. How to solve this problem is the focus of this section. With the aid of some transformation tricks, we recast (P1) as follows.

Proposition 1: Problem (P1) can be converted into an equivalent quadratically constrained quadratic programming

(QCQP) problem (P2) given as follows.

$$\begin{aligned} \max_{\mathbf{y}, v} \quad & \mathbf{b}_0^T \mathbf{y} \\ \text{s.t.} \quad & \mathbf{y}^T \mathbf{A}_0 \mathbf{y} + c_0 v^2 - 1 \leq 0 \\ & \mathbf{y}^T \mathbf{A}_i \mathbf{y} - \mathbf{y}^T \mathbf{b}_i \mathbf{b}_i^T \mathbf{y} + c_i v^2 \leq 0, \quad i = 1, \dots, m_1 \\ & \mathbf{y}^T \mathbf{A}_j \mathbf{y} + c_j v^2 \leq 0, \quad j = m_1 + 1, \dots, m_1 + m_2 \end{aligned} \quad (\text{P2})$$

Proof: The proof is given in Appendix A. ■

It is first noticed that the objective function of (P2) is linear and thus concave. Then to solve (P2), we check the convexity of the feasible set. It is shown that except the ones in the second form, i.e., $\mathbf{y}^T \mathbf{A}_i \mathbf{y} - \mathbf{y}^T \mathbf{b}_i \mathbf{b}_i^T \mathbf{y} + c_i v^2 \leq 0$ for $i = 1, \dots, m_1$, all the others are convex constraints. However, it is found that each constraint in the second form is a union of two convex second-order cone constraints. Consequently, the optimal solution of (P2) can be obtained by solving a collection of convex subproblems. It is concluded in the following proposition.

Proposition 2: Problem (P2) can be efficiently solved by solving a collection of 2^{m_1} SOCP problems.

Proof: The proof is given in Appendix B. ■

It should be pointed out that given the number m_1 , the computational complexity of the approach which solves (P1) is a polynomial function in the dimension of \mathbf{x} .

III. ONE-WAY RELAY CHANNEL

In this section, we consider a two-hop one-way AF relay channel with a single source-destination pair (S, D) and n relay nodes as depicted in Fig. 1. It can be regarded as a layered relay network with three layers. Due to the channel fading, the direct path between the source and destination nodes is neglected. The channel coefficients are all real-valued numbers and remain constant during the operation. Therefore, we only focus on the design of the amplification gains at the relays in different power constraint setups. The full channel state information is revealed to all the network nodes.

Assume that all the relay nodes are equipped with a single antenna and work in a half-duplex mode. Therefore, the data transmission takes place in two steps. During the first step, the source node sends x_S with fixed transmit power P_S . The k th relay receives $y_k = h_{S,k} x_S + z_k$ with z_k being the Gaussian noise and $h_{S,k}$ being the channel coefficient between source node S and relay node k . During the second step, the k th relay sends $x_k = \alpha_k y_k$ with α_k being the amplification gain and hence the received signal at the destination node can be expressed as

$$\begin{aligned} y_D &= \mathbf{h}_2^T \mathbf{A} \mathbf{h}_1 x_S + \mathbf{h}_2^T \mathbf{A} \mathbf{z} + z_D, \\ &= \underbrace{\boldsymbol{\alpha}^T \mathbf{H}_2 \mathbf{h}_1 x_S}_{\text{equivalent source signal}} + \underbrace{\boldsymbol{\alpha}^T \mathbf{H}_2 \mathbf{z} + z_D}_{\text{equivalent Gaussian noise}} \end{aligned} \quad (1)$$

where $\mathbf{A} = \text{diag}\{\alpha_1, \dots, \alpha_n\}$, $\boldsymbol{\alpha} = \text{vec}\{\mathbf{A}\}$, $\mathbf{h}_1 = [h_{S,1}, \dots, h_{S,n}]^T$ and $\mathbf{h}_2 = [h_{1,D}, \dots, h_{n,D}]^T$ denote the channel vectors from the source node to the relays and from the relays to the destination node respectively, $\mathbf{H}_i =$

$\text{diag}\{\mathbf{h}_i\}$, for $i = 1, 2$, and $\mathbf{z} = [z_1, \dots, z_n]^T$ and z_D are the Gaussian noise at the relays and the destination node respectively. The Gaussian noise is independent of the source signal. As the focus of the paper, we consider the Gaussian noise at the relays being correlated which is drawn according to $\mathcal{N}(\mathbf{0}, \mathbf{K})$, but it is independent of $z_D \sim \mathcal{N}(0, \sigma^2)$.

From (1), it follows that the two-hop AF relay channel can be regarded as an equivalent point-to-point Gaussian channel where $\mathbf{h}_2^T \mathbf{A} \mathbf{h}_1 x_S$ is regarded as the source signal and $\mathbf{h}_2^T \mathbf{A} \mathbf{z} + z_D$ is viewed as the Gaussian noise. As is well-known, the source node adopts the Gaussian codebook with x_S drawn according to $\mathcal{N}(0, P_S)$. The achievable rate is given by $R = 0.25 \log(1 + \text{SNR})$, where the pre-log factor is due to the half-duplex assumption and SNR denotes the received signal-to-noise ratio (SNR) at the destination node. Note that $\log(\cdot)$ is an increasing function. Therefore, to maximize the transmission rate is equivalent to maximizing the received SNR at the destination node.

We first consider a general case when the relays are subject to both sum and individual power constraints, i.e., $\alpha^T (P_S \mathbf{H}_1^2 + \mathbf{K} \odot \mathbf{I}) \alpha \leq P_{\text{sum}}$ and $\alpha_k^2 (\sigma_k^2 + h_{S,k}^2 P_S) \leq P_k$ for $k = 1, \dots, n$, where P_{sum} is the sum power budget, P_k is the individual power budget for relay node k , σ_k^2 is the k th diagonal element of \mathbf{K} , and \odot denotes the point-wise multiplication of the two matrix. Then the optimal AF scheme can be obtained via solving the following optimization problem in the form (P1).

$$\begin{aligned} \max_{\alpha} \quad & \text{SNR}(\alpha) = \frac{\alpha^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha P_S}{\alpha^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha + \sigma^2} \\ \text{s.t.} \quad & \alpha^T (P_S \mathbf{H}_1^2 + \mathbf{K} \odot \mathbf{I}) \alpha \leq P_{\text{sum}} \\ & \alpha^T (P_S h_{S,k}^2 + \sigma_k^2) \mathbf{I}_k \alpha \leq P_k, \quad k = 1, \dots, n \end{aligned} \quad (2)$$

where $\mathbf{I}_k = \text{diag}\{0, \dots, 0, 1, 0, \dots, 0\}$ is a diagonal matrix with the k th element equal to 1.

By Proposition 1, it can be recast as

$$\begin{aligned} \max_{\beta, v} \quad & \mathbf{h}_1^T \mathbf{H}_2 \beta \sqrt{P_S} \\ \text{s.t.} \quad & \beta^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \beta + \sigma^2 v^2 - 1 \leq 0 \\ & \beta^T (P_S \mathbf{H}_1^2 + \mathbf{K} \odot \mathbf{I}) \beta - P_{\text{sum}} v^2 \leq 0 \\ & \beta^T (P_S h_{S,k}^2 + \sigma_k^2) \mathbf{I}_k \beta - P_k v^2 \leq 0, \\ & k = 1, \dots, n \end{aligned} \quad (3)$$

Then by Proposition 2, to solve (3), only one SOCP problem should be solved, which is given as follows.

$$\begin{aligned} \max_{\beta, v} \quad & \mathbf{h}_1^T \mathbf{H}_2 \beta \sqrt{P_S} \\ \text{s.t.} \quad & \beta^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \beta + \sigma^2 v^2 - 1 \leq 0 \\ & \|(P_S \mathbf{H}_1^2 + \mathbf{K} \odot \mathbf{I})^{\frac{1}{2}} \beta\|_2 \leq \sqrt{P_{\text{sum}}} v \\ & \|\sqrt{P_S h_{S,k}^2 + \sigma_k^2} \mathbf{I}_k \beta\|_2 \leq \sqrt{P_k} v \\ & k = 1, \dots, n \end{aligned} \quad (4)$$

Denote the optimal solution of (4) by $(\beta_{\text{opt}}, v_{\text{opt}})$, then the optimal solution of (2) is given by $\alpha_{\text{opt}} = \beta_{\text{opt}}/v_{\text{opt}}$.

In particular, if \mathbf{K} is a diagonal matrix, the above problem reduces to the case without relay noise correlation. Thus the proposed approach is also applied to the case. Although the closed-form solution cannot be obtained, we find the proposed approach efficient enough for application in practice.

From the practical perspective, we are especially interested in the case when all the relays are only subject to the individual power constraint. This model can be used to characterize a network with each node having individual power supply. Moreover, to the best of our knowledge, the proposed approach is the most efficient one to solve the problem of finding the optimal AF scheme for the relay network with correlated relay noise and subject to individual relay power constraint.

To further reduce the computational complexity, two suboptimal AF schemes are proposed and the application conditions of them are discussed in the sequel. Given the network parameters, it is easy to compute the following two schemes:

- 1) Suboptimal scheme 1 (for relatively small σ^2):

$$\alpha_1 = c_0 (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2)^{-1} \mathbf{H}_2 \mathbf{h}_1, \quad (5)$$

$$\begin{aligned} \text{where } \alpha_0 &= [\alpha_{0,1}, \dots, \alpha_{0,n}]^T = (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2)^{-1} \mathbf{H}_2 \mathbf{h}_1, \\ \alpha_{k,\max} &= \sqrt{\frac{P_k}{P_S h_{S,k}^2 + \sigma_k^2}}, \text{ for } k = 1, \dots, n, \text{ and } c_0 = \\ &= \min \left\{ \sqrt{\frac{\alpha_{k,\max}^2}{\alpha_{0,k}^2}}, k = 1, \dots, n \right\}. \end{aligned}$$

- 2) Suboptimal scheme 2 (for relatively large σ^2):

$$\alpha_2 = [\text{sign}(h_1) \alpha_{1,\max}, \dots, \text{sign}(h_n) \alpha_{n,\max}]^T, \quad (6)$$

$$\begin{aligned} \text{where } h_k &= h_{S,k} h_{k,D}, \text{ for } k = 1, \dots, n, \text{ and} \\ \text{sign}(h_k) &= 1 \text{ for } h_k \geq 0 \text{ otherwise } \text{sign}(h_k) = -1. \end{aligned}$$

Then we give the following definitions according to α_i , $i = 1, 2$.

Definition 1: We say that the relay noise dominates the sum noise at the destination node if

$$\alpha_1^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_1 \geq \sigma^2, \quad (7)$$

and that the destination noise dominates the sum noise at the destination node if

$$\alpha_2^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_2 \leq \sigma^2, \quad (8)$$

Based on such definitions, we have the following results.

Proposition 3: The achievable rate of α_1 is at most 0.25 bit away from the corresponding optimal one when the relay noise dominates the sum noise, and the achievable rate of α_2 is also at most 0.25 bit away from the corresponding optimal one when the destination noise dominates the sum noise.

Proof: The proof is given in Appendix C. \blacksquare

Finally, we verify our analytical results via the following example.

Example: Let's consider a relay channel with three relay nodes. The network setups are given below:

$$P_S = 1, \sigma^2 = 1, \mathbf{h}_1 = [1, 2, 1]^T, \mathbf{h}_2 = [3, 3, 1]^T,$$

$$\mathbf{K} = \begin{bmatrix} 1.5004 & 1.3293 & 0.8439 \\ 1.3293 & 1.2436 & 0.6936 \\ 0.8439 & 0.6936 & 1.2935 \end{bmatrix}.$$

TABLE I
OPTIMAL AND SUBOPTIMAL AF SCHEMES

P_k	Scheme	α_1	α_2	α_3	$R(\alpha)$
0.1	optimal	0.2000	0.1381	0.2088	0.2421
	suboptimal ₁	-0.1247	0.1318	0.0394	0.0779
	suboptimal ₂	0.2000	0.1381	0.2088	0.2421
5	optimal	-0.8455	0.9765	0.4384	0.8006
	suboptimal ₁	-0.8819	0.9765	0.2788	0.7855
	suboptimal ₂	1.4141	0.9765	1.4765	0.3263

Here we fixed the transmit power of the source node. For brevity, assume that all the relay nodes have the same individual power budget. The optimal and suboptimal AF schemes are shown in table I, where $R(\alpha) = 0.25 \log(1 + \text{SNR}(\alpha))$. From table I, we observe that when $P_k = 0.1$, the corresponding achievable rate of α_2 is optimal and when $P_k = 5$, the corresponding achievable rate of α_1 is approximately optimal. As a consequence, the numerical results coincide with the analysis given above.

So far, we have studied the performance of the optimal and suboptimal AF schemes when the relay noise is correlated. We still curious about the effect of the correlation compared with the independent scenario. Does correlation help? This question has been answered in [10] under sum relay power constraint scenario. However, it seems hard to answer the question under individual relay power constraint scenario due to the lack of analytical optimal AF scheme. Fortunately, it can be handled with the aid of our previous work in [11]. To investigate the issue, we assume that all the channel coefficients are i.i.d Gaussian random variables with zero mean and unit variance. Then we obtain the following results.

Proposition 4: For any source power P_S and relay power constraint P_k , $k = 1, \dots, n$, the AF performance under correlated relay noise is better than or equal to that under independent relay noise in terms of both average SNR and average rate over all the channel realizations.

Proof: The proof is given in Appendix D. ■

IV. TWO-WAY RELAY CHANNEL

In this section, we consider a two-hop two-way AF relay channel consisting of two source nodes S_1 and S_2 and n relay nodes as shown in Fig. 2. We still assume that no direct paths between S_1 and S_2 exists. The uplink channels from S_1 and S_2 to relay node k are denoted as $f_{1,k}$ and $f_{2,k}$ respectively, while $g_{1,k}$ and $g_{2,k}$ denote the downlink channels from relay node k to S_1 and S_2 respectively. All the involved channels are assumed to take real values and remain constant during the operation period. In addition, all the channel state information is revealed to all the network nodes. Again, we assume the relay nodes receive correlated Gaussian noise which is drawn according to $\mathcal{N}(\mathbf{0}, \mathbf{K})$.

To achieve higher spectrum efficiency, a two-step AF protocol is adopted, which is referred to as analog network coding [16]. During the first step, both source nodes simultaneously

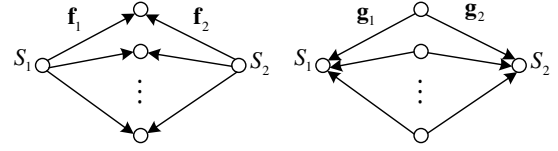


Fig. 2. Two-way Relay Channel

transmit their signals x_{S_1} and x_{S_2} with fixed power P_{S_1} and P_{S_2} respectively to the relay nodes. The k th relay node receives $y_k = f_{1,k}x_{S_1} + f_{2,k}x_{S_2} + z_k$ with z_k being the Gaussian noise. In the second step, the received signal at relay node k is multiplied by the amplification gain α_k and retransmitted. After the self-interference cancellation, the two source nodes receive:

$$y_{0,1} = \underbrace{\mathbf{g}_1^T \mathbf{A} \mathbf{f}_2 x_{S_2}}_{\text{equivalent source signal}} + \underbrace{\mathbf{g}_1^T \mathbf{A} \mathbf{z} + z_{0,1}}_{\text{equivalent Gaussian noise}}, \quad (9)$$

$$y_{0,2} = \underbrace{\mathbf{g}_2^T \mathbf{A} \mathbf{f}_1 x_{S_1}}_{\text{equivalent source signal}} + \underbrace{\mathbf{g}_2^T \mathbf{A} \mathbf{z} + z_{0,2}}_{\text{equivalent Gaussian noise}}, \quad (10)$$

where $\mathbf{z} = [z_1, \dots, z_n]^T$ denotes the noise vector at the relays received in the first step, $\mathbf{A} = \text{diag}\{\alpha_1, \dots, \alpha_n\}$, $z_{0,i}$ is the Gaussian noise with zero mean and variance $\sigma_{0,i}^2$ received at S_i in the second step, $\mathbf{f}_i = [f_{i,1}, \dots, f_{i,n}]^T$ and $\mathbf{g}_i = [g_{i,1}, \dots, g_{i,n}]^T$, for $i = 1, 2$. Assume that the relay noise \mathbf{z} is independent of the source signals and the noise $z_{0,i}$, $i = 1, 2$. Similar to the one-way case, given the amplification gains, the two-way relay channel can be regarded as two point-to-point Gaussian channels. Therefore, both the source nodes adopt Gaussian codebooks, i.e., x_{S_i} is drawn according to $\mathcal{N}(0, P_{S_i})$, for $i = 1, 2$.

The weighted sum rate maximization (WSRM) problem is commonly used to formulate the problem of finding the optimal rate region of a multiuser network. However, for the two-way AF relay channel, we find two major disadvantages of using this formulation. As observed in [19], to solve a WSRM problem with a specific pair of weights, a bisection algorithm is used where each step involves a semi-definite programming (SDP) causing high computational complexity. Furthermore, from the perspective of practical application, to perform AF relaying through this method, every solution corresponding to each pair of weights should be obtained and stored in a look-up table in the control center before the operation period [19]. During normal operations, the control center should look up the table to decide the appropriate scheme that achieves the desirable rate pair. Since the channel coefficients may vary frequently in different operation periods, the whole look-up table should always be updated, which violates real-time communication and consumes much more energy.

To circumvent these disadvantages, we consider an alternative formulation of the problem. The basic idea arises from the fact that to find an approximate scheme that meets the desirable rate pair, we can always regard one transmission rate requirement as a constraint. In addition to the power constraint,

the problem is to maximize the transmission rate of the other one under both the constraints.

Without loss of generality, let's consider the problem of maximizing the transmission rate from S_2 to S_1 under certain power constraint and the constraint of the transmission rate from S_1 to S_2 . As in the one-way case, we equivalently maximize the received SNR at S_1 . The rate constraint is converted into the corresponding SNR constraint at S_2 as well.

We consider a general case with both sum and individual power constraints, i.e., $\alpha^T (P_{S_1} \mathbf{F}_1^2 + P_{S_2} \mathbf{F}_2^2 + \mathbf{K} \odot \mathbf{I}) \alpha \leq P_{sum}$ and $\alpha_k^2 (\sigma_k^2 + f_{1,k}^2 P_{S_1} + f_{2,k}^2 P_{S_2}) \leq P_k$ for $k = 1, \dots, n$. Then the problem is formulated as

$$\begin{aligned} \max_{\alpha} \quad & \frac{\alpha^T \mathbf{G}_1 \mathbf{f}_2 \mathbf{f}_2^T \mathbf{G}_1 \alpha}{\alpha^T \mathbf{G}_1 \mathbf{K} \mathbf{G}_1 \alpha + \sigma_{0,1}^2} P_{S_2} \\ \text{s.t.} \quad & \frac{\alpha^T \mathbf{G}_2 \mathbf{f}_1 \mathbf{f}_1^T \mathbf{G}_2 \alpha}{\alpha^T \mathbf{G}_2 \mathbf{K} \mathbf{G}_2 \alpha + \sigma_{0,2}^2} P_{S_1} \geq \gamma_1 \\ & \alpha^T (P_{S_1} \mathbf{F}_1^2 + P_{S_2} \mathbf{F}_2^2 + \mathbf{K} \odot \mathbf{I}) \alpha \leq P_{sum} \\ & \alpha^T (\sigma_k^2 + f_{1,k}^2 P_{S_1} + f_{2,k}^2 P_{S_2}) \mathbf{I}_k \alpha \leq P_k \\ & k = 1, \dots, n \end{aligned} \quad (11)$$

where $\mathbf{G}_i = \text{diag}\{\mathbf{g}_i\}$, $\mathbf{F}_i = \text{diag}\{\mathbf{f}_i\}$ for $i = 1, 2$, and γ_1 is the equivalent SNR constraint of the rate requirement from S_1 to S_2 .

Then by Proposition 1, it is equivalent to solve

$$\begin{aligned} \max_{\beta, v} \quad & \mathbf{f}_2^T \mathbf{G}_1 \beta \\ \text{s.t.} \quad & \beta^T \mathbf{G}_1 \mathbf{K} \mathbf{G}_1 \beta + \sigma_{0,1}^2 v^2 - 1 \leq 0 \\ & \beta^T \mathbf{A}_1 \beta - \beta^T \mathbf{b}_1 \mathbf{b}_1^T \beta + c_1 v^2 \leq 0 \\ & \beta^T (P_{S_1} \mathbf{F}_1^2 + P_{S_2} \mathbf{F}_2^2 + \mathbf{K} \odot \mathbf{I}) \beta - P_{sum} v^2 \leq 0 \\ & \beta^T (\sigma_k^2 + f_{1,k}^2 P_{S_1} + f_{2,k}^2 P_{S_2}) \mathbf{I}_k \beta - P_k v^2 \leq 0 \\ & k = 1, \dots, n \end{aligned} \quad (12)$$

where $\mathbf{A}_1 = \gamma_1 \mathbf{G}_2 \mathbf{K} \mathbf{G}_2$, $\mathbf{b}_1 = \sqrt{P_{S_1}} \mathbf{G}_2 \mathbf{f}_1$, $c_1 = \gamma_1 \sigma_{0,2}^2$.

Since $m_1 = 1$, by Proposition 2, it follows that $\beta^T \mathbf{A}_1 \beta - \beta^T \mathbf{b}_1 \mathbf{b}_1^T \beta + c_1 v^2 \leq 0$ is separated into two convex second-order cone constraints given by $(\beta^T \mathbf{A}_1 \beta + c_1 v^2)^{\frac{1}{2}} \leq \mathbf{b}_1^T \beta$ and $(\beta^T \mathbf{A}_1 \beta + c_1 v^2)^{\frac{1}{2}} \leq -\mathbf{b}_1^T \beta$. Therefore, two SOCP problems are used to solve (12).

On one hand if the resultant maximum rate satisfy the design requirement, then the relays adopt the corresponding optimal solution during the data exchange. So the scheme is computed instantaneously rather than searching a look-up table. On the other hand, the optimal rate region can also be characterized via such an approach. This goal can be achieved as follows. For the one-way relay channel, the optimal transmission rate can be found and it provides an upper bound on individual rate for the two-way relay channel. It is equivalent to greedily maximizing one transmission rate disregarding the other. For each rate requirement of one user no larger than the value, we may obtain a rate pair which is a boundary point of the union rate region. Then the optimal rate region can be fully characterized after taking the closure of the convex hull of all the derived rate pairs.

V. CONCLUSION

In this paper, we studied both one and two-way AF relay channels. Based on a unified formulation, the optimal AF schemes for both channels are obtained via numerical algorithms efficiently. Our future work is to generalize the existing results to the complex channel scenario.

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APPENDIX A
PROOF OF PROPOSITION 1

Proof: Consider first the following problem.

$$\begin{aligned} \max_{\mathbf{x}, w} \quad & \frac{\mathbf{x}^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}}{w^2} \\ \text{s.t.} \quad & \mathbf{x}^T \mathbf{A}_0 \mathbf{x} + c_0 \leq w^2 \\ & \mathbf{x}^T \mathbf{A}_i \mathbf{x} - \mathbf{x}^T \mathbf{b}_i \mathbf{b}_i^T \mathbf{x} + c_i \leq 0, \quad i = 1, \dots, m_1 \\ & \mathbf{x}^T \mathbf{A}_j \mathbf{x} + c_j \leq 0, \quad j = m_1 + 1, \dots, m_1 + m_2 \end{aligned} \quad (\text{P3})$$

We show that (P1) and (P3) are equivalent. It is observed that the equality should be achieved for the first constraint of (P3) at the optima. This can be proved by contradiction. Let (\mathbf{x}_0, w_0) be an optimal solution of (P3). Suppose $\mathbf{x}_0^T \mathbf{A}_0 \mathbf{x}_0 + c_0 < w_0^2$, it can be found another feasible solution (\mathbf{x}_0, w'_0) with $\mathbf{x}_0^T \mathbf{A}_0 \mathbf{x}_0 + c_0 = w_0'^2$ such that the objective function value of (\mathbf{x}_0, w'_0) is strictly larger than that of $\mathbf{x}_0^T \mathbf{A}_0 \mathbf{x}_0 + c_0 < w_0^2$. It gives a contradiction of the optimality.

Then we prove that the two problems have the same optimal value. On one hand, let \mathbf{x}_0 denote the optimal solution of (P1). It is easy to verify that (\mathbf{x}_0, w_0) where $\mathbf{x}_0^T \mathbf{A}_0 \mathbf{x}_0 + c_0 = w_0^2$, is a feasible solution of (P3). Then we show that (\mathbf{x}_0, w_0) is optimal for (P3). Otherwise, let (\mathbf{x}'_0, w'_0) be an optimal solution of (P3), then we get

$$\begin{aligned} \frac{\mathbf{x}_0^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}_0}{w_0^2} &= \frac{\mathbf{x}_0^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{A}_0 \mathbf{x}_0 + c_0} \\ &> \frac{(\mathbf{x}'_0)^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}'_0}{(\mathbf{x}'_0)^T \mathbf{A}_0 \mathbf{x}'_0 + c_0} = \frac{(\mathbf{x}'_0)^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}'_0}{w_0'^2}, \end{aligned} \quad (13)$$

where the third equality follows from the previous observation, and the inequality and the first equality follows from the assumption. Thus it yields a contradiction of the optimality of (\mathbf{x}'_0, w'_0) .

On the other hand, let (\mathbf{x}_0, w_0) denote the optimal solution of (P3). It is clear that \mathbf{x}_0 satisfies all the constraints in (P1). Then we show that \mathbf{x}_0 is optimal for (P1). Otherwise, let \mathbf{x}'_0 be an optimal solution of (P1) and take $w_0'^2 = (\mathbf{x}'_0)^T \mathbf{A}_0 \mathbf{x}'_0 + c_0$, then we get

$$\begin{aligned} \frac{\mathbf{x}_0^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{A}_0 \mathbf{x}_0 + c_0} &= \frac{\mathbf{x}_0^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}_0}{w_0^2} \\ &> \frac{(\mathbf{x}'_0)^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}'_0}{w_0'^2} = \frac{(\mathbf{x}'_0)^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{x}'_0}{(\mathbf{x}'_0)^T \mathbf{A}_0 \mathbf{x}'_0 + c_0}, \end{aligned} \quad (14)$$

where the first equality follows from the previous observation, and the inequality and the second equality follows from the assumption. Thus it yields a contradiction of the optimality of \mathbf{x}'_0 .

Then we use the trick of variable changing to recast (P3). Let $\mathbf{y} = \mathbf{x}/w$ and $v = 1/w$, we get

$$\begin{aligned} \max_{\mathbf{y}, v} \quad & \mathbf{y}^T \mathbf{b}_0 \mathbf{b}_0^T \mathbf{y} \\ \text{s.t.} \quad & \mathbf{y}^T \mathbf{A}_0 \mathbf{y} + c_0 v^2 - 1 \leq 0 \\ & \mathbf{y}^T \mathbf{A}_i \mathbf{y} - \mathbf{y}^T \mathbf{b}_i \mathbf{b}_i^T \mathbf{y} + c_i v^2 \leq 0, \quad i = 1, \dots, m_1 \\ & \mathbf{y}^T \mathbf{A}_j \mathbf{y} + c_j v^2 \leq 0, \quad j = m_1 + 1, \dots, m_1 + m_2 \end{aligned} \quad (\text{P4})$$

Finally, we prove that to solve (P4), it is sufficient to solve (P2). Denote the optimal solution of (P2) as $(\mathbf{y}_{opt}, v_{opt})$. It is easy to verify that $\mathbf{b}_0^T \mathbf{y}_{opt} \geq 0$ because $(\mathbf{0}, 0)$ is a feasible solution. Suppose it is not the optimal solution of (P4). The rest thing is to show a contradiction. Denote the optimal solution of (P4) as $(\mathbf{y}'_{opt}, v'_{opt})$. Note that (P2) and (P4) have the same feasible region. As a result, $(\mathbf{y}'_{opt}, v'_{opt})$ is also feasible for (P2). By the assumption, it follows that $(\mathbf{b}_0^T \mathbf{y}_{opt})^2 < (\mathbf{b}_0^T \mathbf{y}'_{opt})^2$, which implies either $\mathbf{b}_0^T \mathbf{y}_{opt} < \mathbf{b}_0^T \mathbf{y}'_{opt}$ or $\mathbf{b}_0^T \mathbf{y}_{opt} < -\mathbf{b}_0^T \mathbf{y}'_{opt}$. If the first case holds, then we have obtained the contradiction. Otherwise it is easy to check that $(-\mathbf{y}'_{opt}, v'_{opt})$ is also an optimal solution of (P4) and is feasible for (P2). Consequently, it also yields a contradiction to the optimality of $(\mathbf{y}_{opt}, v_{opt})$.

Then we complete the proof. ■

APPENDIX B
PROOF OF PROPOSITION 2

Proof: It is sufficient to consider $v \geq 0$ when solving (P2). The reason is that only the absolute value of v affects the feasibility of the solutions.

It is observed that the first constraint of (P2) can be rewritten as

$$\begin{bmatrix} \mathbf{y}^T & v \end{bmatrix} \mathbf{Q}_0 \begin{bmatrix} \mathbf{y} \\ v \end{bmatrix} - 1 \leq 0, \quad (15)$$

where $\mathbf{Q}_0 = \text{diag}\{\mathbf{A}_0, c_0\}$. It is a convex quadratic constraint. The second kind of constraint can be regarded as the combination of two parts shown as follows.

$$\|\mathbf{Q}_i^{\frac{1}{2}} \begin{bmatrix} \mathbf{y} \\ v \end{bmatrix}\|_2 \leq \mathbf{b}_i^T \mathbf{y}, \quad (16a)$$

$$\|\mathbf{Q}_i^{\frac{1}{2}} \begin{bmatrix} \mathbf{y} \\ v \end{bmatrix}\|_2 \leq -\mathbf{b}_i^T \mathbf{y}, \quad (16b)$$

where $\mathbf{Q}_i = \text{diag}\{\mathbf{A}_i, c_i\}$ is positive semidefinite for $i = 1, \dots, m_1$. Both of them are convex. Finally consider the third kind of constraint. They can be rewritten as

$$\|\mathbf{A}_j^{\frac{1}{2}} \mathbf{y}\|_2 \leq \sqrt{-c_j} v, \quad \text{for } c_j < 0, \quad (17a)$$

$$\begin{bmatrix} \mathbf{y}^T & v \end{bmatrix} \mathbf{Q}_j \begin{bmatrix} \mathbf{y} \\ v \end{bmatrix} \leq 0, \quad \text{for } c_j \geq 0, \quad (17b)$$

where $\mathbf{Q}_j = \text{diag}\{\mathbf{A}_j, c_j\}$ is positive semidefinite for $j = m_1 + 1, \dots, m_1 + m_2$. Note that (17a) is a second-order cone constraint and (17b) is a convex quadratic constraint.

Since c_j is a constant, either (17a) or (17b) is considered. However, both (16a) and (16b) should be considered. Therefore, to solve (P2), one need to solve totally 2^{m_1} subproblems with each constraint in the second kind either recast as (16a) or (16b). Note that each subproblem is an SOCP problem and thus can be solved in polynomial time. It follows that (P2) can be solved in polynomial time because m_1 is a constant independent of the number of variables. ■

APPENDIX C
PROOF OF PROPOSITION 3

Proof: Consider the case when the relay noise dominates the sum noise at the destination node. We get

$$\begin{aligned}
R(\alpha_1) &= 0.25 \log \left(1 + \frac{\alpha_1^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_1 P_S}{\alpha_1^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_1 + \sigma^2} \right) \\
&\stackrel{(a)}{\geq} 0.25 \log \left(1 + 0.5 \frac{\alpha_1^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_1 P_S}{\alpha_1^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_1} \right) \\
&> 0.25 \log \left(1 + \frac{\alpha_1^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_1 P_S}{\alpha_1^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_1} \right) - 0.25 \\
&\stackrel{(b)}{\geq} R(\alpha_{\text{opt}}) - 0.25,
\end{aligned} \tag{18}$$

where (a) follows from the definition that $\alpha_1^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_1 \geq \sigma^2$, and (b) follows from the fact that α_1 satisfies the individual power constraint and maximizes

$$SNR_{\text{up1}}(\alpha) = \frac{\alpha^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha P_S}{\alpha^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha}.$$

It is easy to verify that $SNR_{\text{up1}}(\alpha_1)$ provides an upper bound to $SNR(\alpha_{\text{opt}})$.

Next, we consider the case when the destination noise dominates the sum noise at the destination node.

$$\begin{aligned}
R(\alpha_2) &= 0.25 \log \left(1 + \frac{\alpha_2^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_2 P_S}{\alpha_2^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_2 + \sigma^2} \right) \\
&\stackrel{(a)}{\geq} 0.25 \log \left(1 + 0.5 \frac{\alpha_2^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_2 P_S}{\sigma^2} \right) \\
&> 0.25 \log \left(1 + \frac{\alpha_2^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_2 P_S}{\sigma^2} \right) - 0.25 \\
&\stackrel{(b)}{\geq} R(\alpha_{\text{opt}}) - 0.25,
\end{aligned} \tag{19}$$

where (a) follows from the definition that $\alpha_2^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_2 \leq \sigma^2$, and (b) follows from the fact that α_2 satisfies the individual power constraint and maximizes

$$SNR_{\text{up2}}(\alpha) = \frac{\alpha^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha P_S}{\sigma^2}.$$

It is easy to verify that $SNR_{\text{up2}}(\alpha_2)$ provides an upper bound to $SNR(\alpha_{\text{opt}})$.

Then we complete the proof. \blacksquare

APPENDIX D
PROOF OF PROPOSITION 4

Proof: For fairness in comparison, we assume that the marginal distribution of the relay noise is the same for all the scenarios. Thus if \mathbf{K} is a general noise covariance matrix, then $\mathbf{K} \odot \mathbf{I}$ is the noise covariance when noise is uncorrelated. To prove the proposition, we compare the following scenarios.

- 1) *Scenario a:* The relay noise is correlated and relays are aware of the correlation. Therefore, the relays adopt the optimal AF scheme obtained above. Denote the corresponding SNR value by $SNR_a = SNR(\alpha_{\text{opt}})$.
- 2) *Scenario b:* The noise is uncorrelated. Therefore, the relays adopt the optimal AF scheme obtained in [11]

denoted as α_0 , which is given in the sequel. The corresponding SNR value is denoted by $SNR_b = SNR_0(\alpha_0)$, where

$$SNR_0(\alpha) = \frac{\alpha^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha P_S}{\alpha^T (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \odot \mathbf{I}) \alpha + \sigma^2}.$$

- 3) *Scenario c:* The noise is correlated and relays are unaware of the correlation. In such a scenario, the relays adopt α_0 . The corresponding SNR value is denoted by $SNR_c = SNR(\alpha_0)$.

If the performance of α_{opt} under correlated relay noise is better than the performance of α_0 under independent relay noise, then we say that correlation helps if the relays are aware of it. Similarly, if α_0 has better performance under independent relay noise than under correlated relay noise, then we can say that correlation hurts when the relays are unaware of it.

We first revisit α_0 as follows. As observed in [11], after finding the optimal active sets \mathcal{M}^* , the optimal AF scheme is

$$\alpha_0 = c (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \odot \mathbf{I} + \sigma^2 \mathbf{G})^{-1} \mathbf{H}_2 \mathbf{h}_1, \tag{20}$$

where $\mathbf{G} = \text{diag}\{g_1, \dots, g_n\}$, $g_j = 0$, $j \notin \mathcal{M}^*$, g_i , $i \in \mathcal{M}^*$ and c are given as follows.

$$\begin{aligned}
c &= \frac{\sum_{i \in \mathcal{M}^*} \alpha_{i,\max}^2 h_{i,D}^2 \sigma_i^2 + \sigma^2}{\sum_{i \in \mathcal{M}^*} \alpha_{i,\max} |h_{i,D} h_{S,i}|}, \\
g_i &= c \frac{|h_{i,D} h_{S,i}|}{\alpha_{i,\max} \sigma^2} - \frac{h_{i,D}^2 \sigma_i^2}{\sigma^2}, \quad i \in \mathcal{M}^*.
\end{aligned}$$

The corresponding SNR value of α_0 under independent noise is given as

$$\begin{aligned}
SNR_b &= SNR_0(\alpha_0) \\
&= \mathbf{h}_1^T \mathbf{H}_2 (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \odot \mathbf{I} + \sigma^2 \mathbf{G})^{-1} \mathbf{H}_2 \mathbf{h}_1 P_S,
\end{aligned} \tag{21}$$

It should be noticed that the choice of the optimal active set \mathcal{M}^* depends on the network parameters including channel coefficients \mathbf{h}_1 , \mathbf{h}_2 and transmit power constraints P_S and P_k , for $k = 1, \dots, n$. Nevertheless, it can be easily shown that given the amplitudes of all the channel coefficients, \mathcal{M}^* remains constant while changing the signs of them. It is because in the absence of correlation, \mathbf{K} is replaced by $\mathbf{K} \odot \mathbf{I}$ and thus diagonal. So, the sign of the amplification gain at the relay node k is always chosen such that it coincides with $\text{sign}(h_{S,k} h_{k,D})$ and the noise power remains unchanged. Therefore, the sign of channel coefficient does not affect the choice of the optimal active set. Then we get

$$\begin{aligned}
SNR_c &= \frac{\alpha_0^T \mathbf{H}_2 \mathbf{h}_1 \mathbf{h}_1^T \mathbf{H}_2 \alpha_0 P_S}{\alpha_0^T \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \alpha_0 + \sigma^2} \\
&= \frac{a SNR_b}{a + b},
\end{aligned} \tag{22}$$

where $a = \alpha_0^T (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \odot \mathbf{I}) \alpha_0 + \sigma^2$ and

$$\begin{aligned} b &= \alpha_0^T (\mathbf{H}_2 \mathbf{K} \mathbf{H}_2 - \mathbf{H}_2 \mathbf{K} \mathbf{H}_2 \odot \mathbf{I}) \alpha_0 \\ &= c^2 \sum_{i=1}^N \sum_{j \neq i}^N h_{S,i} h_{S,j} \frac{h_{i,D}^2 K(i,j) h_{j,D}^2}{\left(\sigma_i^2 h_{i,D}^2 + \sigma^2 g_i\right) \left(\sigma_j^2 h_{j,D}^2 + \sigma^2 g_j\right)}. \end{aligned} \quad (23)$$

It is easy to check that the second derivative of SNR_c with respect to b is $\frac{2aSNR_b}{(a+b)^3}$ which is always positive since a , $(a+b)$ and SNR_b are always positive. Thus, SNR_c is convex with respect to b . Since we have assumed that all the channel coefficients are i.i.d Gaussian, $\text{sign}(h_{S,i})$ and $\text{sign}(h_{S,j})$ are independent and take values 1 and -1 with equal probability. Then from (23), it follows that the expected value of b over all signs of the channel coefficients is 0 conditioned on the absolute values of the channel coefficients, while the values of a and SNR_b remains constant. Now consider the expected value of SNR_c

$$\begin{aligned} \mathbb{E}[SNR_c] &= \mathbb{E} \mathbb{E}_{\text{sign}} \left[\frac{aSNR_b}{a+b} \right] \\ &\stackrel{(a)}{\geq} \mathbb{E} \left[\frac{aSNR_b}{a + \mathbb{E}_{\text{sign}}[b]} \right] \\ &= \mathbb{E}[SNR_b] \end{aligned} \quad (24)$$

where (a) follows from Jensen's inequality.

Let us now consider $R_c = 0.25 \log \left(1 + \frac{aSNR_b}{a+b} \right)$, the achievable rate of α_0 under correlated relay noise. It can be shown that the second derivative of R_c with respect to b is $\frac{1}{4 \ln 2} \frac{aSNR_b(2(a+b)+aSNR_b)}{(a+b+aSNR_b)^2(a+b)^2}$ which is always positive. Therefore, R_c is convex in b . Now consider the average rate $\mathbb{E}[R_c]$

$$\begin{aligned} \mathbb{E}[R_c] &= \mathbb{E} \mathbb{E}_{\text{sign}} \left[0.25 \log \left(1 + \frac{aSNR_b}{a+b} \right) \right] \\ &\stackrel{(a)}{\geq} \mathbb{E} \left[0.25 \log \left(1 + \frac{aSNR_b}{a + \mathbb{E}_{\text{sign}}[b]} \right) \right] \\ &= \mathbb{E}[0.25 \log(1 + SNR_b)] = \mathbb{E}[R_b]. \end{aligned} \quad (25)$$

Then we complete the proof. \blacksquare